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INFINITE FAMILIES OF ISOMORPHIC NONCONJUGATE FINITELY GENERATED SUBGROUPS

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ABSTRACT. Let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L which splits as an orthogonal direct sum $(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle) \perp (L_3, \langle\ ,\ \rangle)$ in which $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, $(L_2, \langle\ ,\ \rangle)$ has signature $(n, 1)$ with $n \geq 2$, and $(L_3, \langle\ ,\ \rangle)$ is either zero or indefinite with $\mathrm{rk}_{\mathbf{Z}}(L_3) \geq 3$. We show that the integral automorphism group $\mathrm{Aut}_{\mathbf{Z}}(L, \langle\ ,\ \rangle)$ contains an infinite family of mutually isomorphic finitely generated subgroups $(\Gamma_\sigma)_{\sigma\in\Sigma}$, no two of which are conjugate. In the simplest case, when $L_3 = 0$, the groups Γ_σ are all normal subdirect products in a product of free groups or surface groups. The result can be seen as a failure of the rigidity property for subgroups of infinite covolume within the corresponding Lie group $\mathrm{Aut}_{\mathbf{Z}}(L \otimes_{\mathbf{Z}} \mathbb{R}, \langle\ ,\ \rangle \otimes 1)$.

0. INTRODUCTION

The following question arose from the joint work of Ebeling and Okonek on diffeomorphisms of algebraic surfaces.

Question. Let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L. When, if ever, does there exist an infinite family of isomorphic finitely generated subgroups $(\Gamma_\sigma)_{\sigma\in\Sigma}$ of $\mathrm{Aut}_{\mathbf{Z}}(L, \langle\ ,\ \rangle)$ such that for $\sigma \neq \tau$, Γ_σ is not conjugate to Γ_τ in $\mathrm{Aut}_{\mathbf{Z}}(L, \langle\ ,\ \rangle)$?

In this paper, we establish the existence of such infinite families $(\Gamma_\sigma)_{\sigma\in\Sigma}$ of nonconjugate isomorphic finitely generated subgroups when $(L, \langle\ ,\ \rangle)$ splits as an orthogonal direct sum

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle) \perp (L_3, \langle\ ,\ \rangle)$$

in which $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, $(L_2, \langle\ ,\ \rangle)$ has signature $(n, 1)$ with $n \geq 2$, and $(L_3, \langle\ ,\ \rangle)$ is either zero or indefinite with $\mathrm{rk}_{\mathbf{Z}}(L_3) \geq 3$. The parameter set Σ may be thought of as an infinite subset of $\mathrm{Aut}_{\mathbf{Z}}(L, \langle\ ,\ \rangle)$.

The construction of the groups $(\Gamma_\sigma)_{\sigma\in\Sigma}$ uses a variation on the methods of our earlier paper [3]; in addition, the main theorem of [3] is needed to show finite generation. In §1, we recall some basic facts about orthogonal groups and integral quadratic forms. The necessary results from [3] are reviewed in §§2–3, and the families $(\Gamma_\sigma)_{\sigma\in\Sigma}$ are constructed in §4 (Theorems 4.4 and 4.5).

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1. Integral quadratic forms and their arithmetic subgroups

Let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric integral bilinear form on a free abelian group L of finite rank m, say. $(L, \langle\ ,\ \rangle)$ is said to be *isotropic* (*over* $\mathbb{Z}$) when there exists a nonzero element $\mathbf{x} \in L$ such that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$; otherwise $(L, \langle\ ,\ \rangle)$ is said to be *anisotropic.* Put $\Gamma = \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$. The associated real form $\langle\ ,\ \rangle : L \otimes \mathbb{R} \times L \otimes \mathbb{R} \to \mathbb{R}$ is diagonalisable as

$$\sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{p+q} x_i y_i ,$$

assigning to $(L, \langle\ ,\ \rangle)$ the signature (p, q) where $p + q = m$; Γ imbeds as a discrete subgroup of finite covolume in the group $\mathrm{Aut}_{\mathbb{R}}(L \otimes \mathbb{R}, \langle\ ,\ \rangle) \cong O(p, q)$, and acts properly discontinuously as a group of isometries of the symmetric space of $O(p, q)$. Moreover, Γ is cocompact precisely when $\langle\ ,\ \rangle$ is anisotropic. (When $L, \langle\ ,\ \rangle)$ is indefinite, a classical theorem of Meyer [5] asserts that for $\langle\ ,\ \rangle$ to be anisotropic it is necessary that $m \leq 4$.)

When the signature of $(L, \langle\ ,\ \rangle)$ is $(2, 1)$, the corresponding symmetric space is the upper half-plane, so that Γ is a Fuchsian group. When $(L, \langle\ ,\ \rangle)$ is isotropic, Γ contains a nonabelian free subgroup of finite index. When $(L, \langle\ ,\ \rangle)$ is anisotropic, Γ contains, as a subgroup of finite index, a surface group Σ_g^+; that is, the fundamental group of an orientable surface of genus $g \geq 2$, having a presentation of the form

$$\Sigma_g^+ = \left\langle X_1, \dots, X_g, Y_1, \dots, Y_g : \prod_{r=1}^{g} X_r Y_r X_r^{-1} Y_r^{-1} \right\rangle .$$

We summarise these observations.

Proposition 1.1. *Let* Γ *be the automorphism group of a nondegenerate integral quadratic form of signature* $(2, 1)$; *then* Γ *is finitely generated, and*

(i) Γ *contains a surface subgroup of finite index when* $(L, \langle\ ,\ \rangle)$ *is anisotropic;*

(ii) Γ *contains a nonabelian free subgroup of finite index when* $(L, \langle\ ,\ \rangle)$ *is isotropic.*

Let $\mathbb{G}$ be a linear algebraic group defined and semisimple over $\mathbb{Q}$; we may take $\mathbb{G}$ to be imbedded $\mathbb{G}_{\mathbb{Q}} \subset \mathbb{GL}_n(\mathbb{Q})$. By an *arithmetic subgroup* of $\mathbb{G}$, we mean a subgroup Γ of $\mathbb{G}_{\mathbb{R}}$ which is commensurable with $\mathbb{G}_{\mathbb{Z}} = \mathbb{G}_{\mathbb{Q}} \cap \mathbb{GL}_n(\mathbb{Z})$. This does not depend on the particular imbedding $\mathbb{G}_{\mathbb{Q}} \subset \mathbb{GL}_n(\mathbb{Q})$ chosen. Moreover, for such a subgroup Γ, $\mathbb{G}_{\mathbb{R}}/\Gamma$ has finite invariant volume. Let $\overline{\Delta} \subset \mathbb{G}_{\mathbb{C}}$ denote the Zariski closure of a subgroup $\Delta \subset \mathbb{G}_{\mathbb{R}}$.

We begin by observing the following, where $[\Gamma, \Gamma]$ denotes the commutator subgroup of Γ.

Proposition 1.2. *Let* $\mathbb{G}$ *be a linear algebraic group defined and semisimple over* $\mathbb{Q}$, *with the property that* $\mathbb{G}_{i,\mathbb{R}}$ *is noncompact for each* $\mathbb{Q}$*-simple factor* $\mathbb{G}_i$. *If* Γ *is an arithmetic subgroup of* $\mathbb{G}$ *then* $\overline{[\Gamma, \Gamma]} = \mathbb{G}_{\mathbb{C}}$.

Proof. We first consider the case where $\mathbb{G}$ is $\mathbb{Q}$-simple. By Borel's Density Theorem in the form of [1], $\overline{\Gamma} = \mathbb{G}_{\mathbb{C}}$, and since $\mathbb{G}_{\mathbb{C}}$ is nonabelian, Γ is also nonabelian; hence $[\Gamma, \Gamma]$ is nontrivial. Γ normalises $[\Gamma, \Gamma]$, so that $\overline{\Gamma}$ normalises $\overline{[\Gamma, \Gamma]}$. However, since $\overline{\Gamma} = \mathbb{G}_{\mathbb{C}}$, $\overline{[\Gamma, \Gamma]}$ is a *normal* complex algebraic subgroup of $\mathbb{G}_{\mathbb{C}}$. Moreover, since $\overline{[\Gamma, \Gamma]}$ is the Zariski closure of a subset $[\Gamma, \Gamma]$ of $\mathbb{G}_{\mathbb{Q}}$, then by Weil's Rationality Criterion [8], $\overline{[\Gamma, \Gamma]}$ is defined over $\mathbb{Q}$. The assertion that $\overline{[\Gamma, \Gamma]} = \mathbb{G}_{\mathbb{C}}$ now follows from the fact that $\mathbb{G}$ is $\mathbb{Q}$-simple and $\overline{[\Gamma, \Gamma]}$ is nontrivial.

In general, $\mathbb{G}$ is isogenous with the product of its $\mathbb{Q}$-simple factors $\mathbb{G}_1 \times \cdots \times \mathbb{G}_n$, so that Γ contains, with finite index, a subgroup of the form $\Gamma_1 \times \cdots \times \Gamma_n$, where Γ_i is an arithmetic subgroup of $\mathbb{G}_i$. Hence $[\Gamma_1, \Gamma_1] \times \cdots \times [\Gamma_n, \Gamma_n]$ is contained in $[\Gamma, \Gamma]$, and the result follows easily from the special case already considered. □

For any field k, let $O(n, k)$ denote the group of automorphisms of the standard symmetric bilinear form

$$\langle\ ,\ \rangle : k^n \times k^n \to k; \qquad \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i,$$

and let $\mathfrak{O}(n, k)$ denote the Lie algebra of $O(n, k)$,

$$\mathfrak{O}(n, k) = \{A \in M_n(k) : A^T + A = 0\}.$$

The obvious isomorphism $k^{n_1} \oplus \cdots \oplus k^{n_f} \cong k^{n_1 + \cdots + n_f}$ induces injections

$$\mathfrak{O}(n_1, k) \oplus \cdots \oplus \mathfrak{O}(n_f, k) \subset \mathfrak{O}(n_1 + \cdots + n_f, k),$$

and

$$O(n_1, k) \times \cdots \times O(n_f, k) \subset O(n_1 + \cdots + n_f, k).$$

Proposition 1.3. $\mathfrak{O}(n_1, \mathbb{C}) \oplus \cdots \oplus \mathfrak{O}(n_f, \mathbb{C})$ *is a self-normalising Lie subalgebra of* $\mathfrak{O}(n_1 + \cdots + n_f, \mathbf{C})$ *provided that each* $n_i \geq 2$.

Proof. It clearly suffices to show that $\mathfrak{O}(m, \mathbb{C}) \oplus \mathfrak{O}(n, \mathbb{C})$ is a self-normalizing Lie subalgebra of $\mathfrak{O}(m + n, \mathbb{C})$ provided that $m, n \geq 2$; the general case follows easily by induction. Thus suppose that $\alpha \in M_{m+n}(\mathbb{C})$ has the property

$$(*) \qquad [\alpha, \xi] \in \mathfrak{O}(m, \mathbb{C}) \oplus \mathfrak{O}(n, \mathbb{C}) \quad \text{for all } \xi \in \mathfrak{O}(m, \mathbb{C}) \oplus \mathfrak{O}(n, \mathbb{C}).$$

We may write α, ξ in block form:

$$\alpha = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \xi = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

where $X \in \mathfrak{O}(m, \mathbb{C})$ and $Y \in \mathfrak{O}(n, \mathbb{C})$ so that

$$[\alpha, \xi] = \begin{bmatrix} [A, X] & BY - XB \\ CX - YC & [D, Y] \end{bmatrix}.$$

The condition that $[\alpha, \xi] \in \mathfrak{O}(m, \mathbb{C}) \oplus \mathfrak{O}(n, \mathbb{C})$ implies that $BY - XB = 0$ and $CX - YC = 0$. However, if $BY - XB = 0$ for all $X \in \mathfrak{O}(m, \mathbb{C})$ and all $Y \in \mathfrak{O}(n, \mathbb{C})$, then we may take Y to be the zero matrix and, since $m \geq 2$,

X to be an invertible skew-symmetric matrix, from which we see immediately that $B = 0$. Similarly $C = 0$. If we now impose the additional condition that $\alpha \in \mathfrak{O}(m+n, \mathbb{C})$, that is, $\alpha^T + \alpha = 0$, we see that $A^T + A = 0$ and $D^T + D = 0$. Hence $\alpha \in \mathfrak{O}(m, \mathbb{C}) \oplus \mathfrak{O}(n, \mathbb{C})$ as claimed. □

For any group G and subgroup H, we denote by $N_G(H)$ the normaliser of H in G. When $\mathbb{G}$ is an algebraic group and $\mathbb{H}$ is an algebraic subgroup, $N_{\mathbb{G}}(\mathbb{H})$ is also an algebraic subgroup of $\mathbb{G}$. In particular, the normaliser $N(n_1, \dots, n_f)$ of $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ in $O(n_1 + \cdots + n_f, \mathbb{C})$, is an algebraic subgroup of $O(n_1 + \cdots + n_f, \mathbb{C})$. It follows that $N(n_1, \dots, n_f)$ is a complex Lie group; moreover, when each $n_i \geq 2$, it follows from Proposition 1.3 that $N(n_1, \dots, n_f)$ has the same identity component as $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$. Since any linear algebraic group over $\mathbb{C}$ has only finitely many connected components [2, p. 86]), we see that

Corollary 1.4. *Let $N(n_1, \dots, n_f)$ be the normaliser of $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ in $O(n_1 + \cdots + f_n, \mathbb{C})$; then $O(n_1, \mathbb{C}) \times \cdots \times O(n_f, \mathbb{C})$ has finite index in $N(n_1, \dots, n_f)$ provided that each $n_i \geq 2$.*

Proposition 1.5. *Let L be a finitely generated free abelian group, and let $\langle\ ,\ \rangle: L \times L \to \mathbb{Z}$ be a nondegenerate symmetric integral bilinear form which splits as a direct sum*

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle) \perp \cdots \perp (L_f, \langle\ ,\ \rangle)$$

where $f \geq 2$, and each $\mathrm{rk}_{\mathbb{Z}}(L_i) \geq 2$. Let $\mathbb{G}$ (resp. $\mathbb{G}_i$) be the linear algebraic group whose group of k-rational points is $\mathrm{Aut}_k(L \otimes k, \langle\ ,\ \rangle)$, (resp. $\mathrm{Aut}_k(L_i \otimes k, \langle\ ,\ \rangle)$), and let

$$\mathbb{H} = \mathbb{G}_1 \times \cdots \times \mathbb{G}_f \subset \mathbb{G};$$

then $N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ contains $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \cdots \times \mathrm{Aut}_{\mathbb{Z}}(L_f, \langle\ ,\ \rangle)$ as a subgroup of finite index.

Proof. Put $\lambda_i = \mathrm{rk}_{\mathbb{Z}}(L_i)$, and $\lambda = \sum \lambda_i$. $\mathbb{H}$ and $N_{\mathbb{G}}(\mathbb{H})$ are both linear algebraic subgroups of $\mathbb{G}$, defined over $\mathbb{Q}$, and the groups of real points, $\mathbb{H}_{\mathbf{R}}$ and $(N_{\mathbb{G}}(\mathbb{H}))_{\mathbf{R}}$ respectively, are Lie groups possessing only finitely many connected components. Observe that $\mathbb{G}_{\mathbb{C}}$ (respectively $\mathbb{G}_{i,\mathbb{C}}$) is isomorphic to $O(\lambda, \mathbb{C})$ (respectively $O(\lambda_i, \mathbb{C})$), so that, by Corollary 1.4, $\mathbb{H}_{\mathbb{C}}$ is a subgroup of finite index in $(N_{\mathbb{G}}(\mathbb{H}))_{\mathbb{C}}$. Thus the identity components of the corresponding real groups are equal; that is, $\mathbb{H}_{\mathbf{R},0} = (N_{\mathbb{G}}(\mathbb{H}))_{\mathbf{R},0}$. The conclusion follows since $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \cdots \times \mathrm{Aut}_{\mathbb{Z}}(L_f, \langle\ ,\ \rangle)$ and $N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ are both arithmetic in $N_{\mathbb{G}}(\mathbb{H})$, and $N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ contains $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \cdots \times \mathrm{Aut}_{\mathbb{Z}}(L_f, \langle\ ,\ \rangle)$. □

2. Normal subdirect products

By a *product structure* on a group G we mean a finite sequence $\mathcal{G} = (G_r)_{1 \leq r \leq n}$ of (nontrivial) normal subgroups of G such that G is the internal direct product $G = G_1 \circ \cdots \circ G_n$; that is, each $g \in G$ can be expressed uniquely as a product $g = g_1 \cdots g_n$ with $g_i \in G_i$. For a group G having a product structure $\mathcal{G} = (G_r)_{1 \leq r \leq n}$, we identify G with the *external direct product* $\prod_{j=1}^n G_j$. Let $\pi_i: \prod_{j=1}^n G_j \to G_i$ be the projection onto the ith factor; a subgroup H

of $\prod_{i=1}^{n} G_i$ is a *subdirect product* of G (or more accurately, of $\mathscr{G}$) when $\pi_i(H) = G_i$ for each i. Let $S(G_1, \dots, G_n)$ the set of *normal subdirect products* of $G_1 \circ \cdots \circ G_n$; that is, subdirect products which are also normal subgroups.

For any group H, let $\nu \colon H \to H^{\mathrm{ab}}$ denote the canonical map onto the abelianisation $H^{\mathrm{ab}} = H/[H, H]$. To any product structure $\mathscr{G} = (G_r)_{1 \le r \le n}$, we may associate its abelianisation $\mathscr{G}^{\mathrm{ab}} = (G_r^{\mathrm{ab}})_{1 \le r \le n}$. Moreover, the abelianisation map $\nu \colon G_1 \circ \cdots \circ G_n \to G_1^{\mathrm{ab}} \circ \cdots \circ G_n^{\mathrm{ab}}$ induces a mapping

$$\nu^{-1} \colon S(G_1^{\mathrm{ab}}, \dots, G_k^{\mathrm{ab}}) \to S(G_1, \dots, G_k)$$

by means of $H \mapsto \nu^{-1}(H)$. We have shown elsewhere [3, Proposition 1.2] that

Proposition 2.1. *For any product structure* $\mathscr{G} = (G_r)_{1 \le r \le n}$

$$\nu^{-1} \colon S(G_1^{\mathrm{ab}}, \dots, G_n^{\mathrm{ab}}) \to S(G_1, \dots, G_n)$$

is bijective.

The following result of [3, Corollary 3.6] is important in the sequel.

Theorem 2.2. *Let H be a normal subdirect product of $G_1 \circ \cdots \circ G_n$. Then H is finitely generated (as a group, not merely as a normal subgroup) if and only if each G_i is finitely generated.*

The conclusion of Theorem 2.2 is false if the assumption of normality on H is dropped.

3. A construction for abelian subdirect products

Let B be an infinite finitely generated abelian group. By an *oriented splitting* for B, we shall mean a triple X of the form $X = (M_X, N_X, \varepsilon_X)$, where $B/\mathrm{Tor}(B) = M_X \oplus N_X$ in which N_X is free of rank 1, and $\varepsilon_X \in N_X$ is a generator. We denote by $\mathfrak{S}(B)$ the set of oriented splittings of B. Clearly the group $\mathrm{Aut}(B/\mathrm{Tor}(B))$ acts transitively on $\mathfrak{S}(B)$. Since $\mathrm{Tor}(B)$ is a characteristic subgroup of B, there is a natural epimorphism $\mathrm{Aut}(B) \to \mathrm{Aut}(B/\mathrm{Tor}(B))$, from which we see that $\mathrm{Aut}(B)$ also acts transitively on $\mathfrak{S}(B)$.

We now consider subdirect products of abelian groups; it is more convenient to write our groups additively, and to confuse direct products with direct sums. Thus suppose that $A = A_1 \oplus A_2$ where A_1 is a free abelian group of rank $r_1 \ge 2$, and A_2 is a finitely generated abelian group such that $A_2/\mathrm{Tor}(A_2)$ has rank $r_2 \ge 1$.

Let $X = (M_X, N_X, \varepsilon_X)$ be an oriented splitting for A_1, and $Y = (M_Y, N_Y, \varepsilon_Y)$ an oriented splitting for $A_2/\mathrm{Tor}(A_2)$. Let $\delta(X, Y)$ denote the subgroup of $A_1 \oplus A_2/\mathrm{Tor}(A_2)$ defined by

$$\delta(X, Y) = M_X \oplus \langle \varepsilon_X + \varepsilon_Y \rangle \oplus M_Y,$$

and let $\Delta(X, Y)$ denote the preimage of $\delta(X, Y)$ in $A = A_1 \oplus A_2$, under the natural mapping

$$\psi \colon A_1 \oplus A_2 \to A_1 \oplus A_2/\mathrm{Tor}(A_2).$$

It is easy to see that $\Delta(X, Y)$ is a (necessarily normal) subdirect product of $A_1 \oplus A_2$. The group $\mathrm{Aut}(A_1, A_2)$ of product preserving automorphisms of $A_1 \oplus A_2$ acts naturally on $S(A_1, A_2)$. Since $\mathrm{Aut}(A_1)$ imbeds naturally in $\mathrm{Aut}(A_1, A_2)$, by extending its natural action on A_1 via the identity on A_2, we see that

$\mathrm{Aut}(A_1)$ also acts naturally on $S(A_1, A_2)$. On taking $\Delta = \Delta(X, Y)$ for some suitable oriented splittings $X = (M_X, N_X, \varepsilon_X)$ and $Y = (M_Y, N_Y, \varepsilon_Y)$ for A_1 and $A_2/\mathrm{Tor}(A_2)$ respectively, we obtain

Theorem 3.1. *Let A_1, A_2 be finitely generated abelian groups such that A_1 is free abelian of rank $r_1 \geq 2$, and $A_2/\mathrm{Tor}(A_2)$ has rank $r_2 \geq 1$. Then there is a subdirect product $\Delta \subset A_1 \oplus A_2$, and an infinite subset $\Theta \subset \mathrm{Aut}(A_1)$ such that $\theta(\Delta) \neq \sigma(\Delta)$ for $\theta, \sigma \in \Theta$ such that $\theta \neq \sigma$.*

4. Infinite families of nonconjugate isomorphic imbeddings

Let Λ_1 be a nonabelian free group of finite rank $m \geq 2$, and let Λ_2 be a finitely generated group such that Λ_2^{ab} is infinite. Put $A_i = \Lambda_i^{\mathrm{ab}}$ for $i = 1, 2$. Since $A_1 \cong \mathbb{Z}^m$ and A_2 maps epimorphically onto $\mathbb{Z}$, we may apply Theorem 3.1 to obtain the existence of a faithfully indexed family $(\theta(\Delta))_{\theta\in\Theta}$ of normal subdirect products of $A_1 \oplus A_2$, where θ ranges over some infinite subset Θ of $\mathrm{Aut}(A_1) \cong \mathrm{GL}_m(\mathbb{Z})$. As we have seen, $\nu^{-1}\colon S(A_1^{\mathrm{ab}}, A_2^{\mathrm{ab}}) \to S(\Lambda_1, \Lambda_2)$ is bijective. Put $\Gamma = \nu^{-1}(\Delta)$; then Γ is a normal subdirect product of $\Lambda_1 \times \Lambda_2$, and so is finitely generated by Theorem 2.2. Furthermore, the group $\mathrm{Aut}(\Lambda_1) \times \mathrm{Aut}(\Lambda_2)$ acts naturally on subgroups of $\Lambda_1 \times \Lambda_2$, and the orbit of Γ under this action consists entirely of normal subdirect products of $\Lambda_1 \times \Lambda_2$. In fact, we need only consider the subgroup $\mathrm{Aut}(\Lambda_1) \cong \mathrm{Aut}(\Lambda_1) \times \{1\}$ of $\mathrm{Aut}(\Lambda_1) \times \mathrm{Aut}(\Lambda_2)$. Since Λ_1 is free, by a theorem of Nielsen [7], for each automorphism θ of $A_1 = \Lambda_1^{\mathrm{ab}}$ we may choose a lifting of θ to an automorphism $\hat{\theta}$ of $\Lambda_1 \cong \Lambda_1 \times \{1\}$. Put $\Gamma_\theta = \hat{\theta}(\Gamma)$. It is clear that Γ_θ is isomorphic to Γ. We may summarise our progress so far thus:

Theorem 4.1. *Let Λ_1 be a nonabelian free group of finite rank $m \geq 2$, and let Λ_2 be a finitely generated group which maps epimorphically onto $\mathbb{Z}$; then there is a subset $\Theta \subset \mathrm{Aut}(A_1)$ which parametrises an infinite family $(\Gamma_\theta)_{\theta\in\Theta}$ of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$ with the property that $\Gamma_\theta \neq \Gamma_\sigma$ for $\theta \neq \sigma$.*

The analogue of Theorem 4.1 in which Λ_1 is replaced by the fundamental group of a closed orientable surface is also true; we proceed to outline the necessary variations.

Let Σ_+^g denote the closed orientable surface of genus $g \geq 2$, and let Σ_g^+ denote its fundamental group;

$$\Sigma_g^+ = \left\langle X_1, \dots, X_g, Y_1, \dots, Y_g \colon \prod_{r=1}^{g} X_r Y_r X_r^{-1} Y_r^{-1} \right\rangle .$$

We may identify the abelianisation $H_1(\Sigma_g^+; \mathbb{Z})$ of Σ_g^+ with $\mathbb{Z}^{2g}$, and the intersection form on Σ_+^g gives rise to a nondegenerate skew-symmetric bilinear form $\langle\ ,\ \rangle\colon \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$. With this identification, *symplectic automorphisms* of $\mathbb{Z}^{2g}$ lift back to automorphisms of $\Sigma_g^+ = \pi_1(\Sigma_+^g)$, with transvections lifting back to Dehn twists.

Let $\{\varepsilon_1, \dots, \varepsilon_g, \phi_1, \dots, \phi_g\}$ be the standard symplectic basis for $\langle\ ,\ \rangle$; that is,

$$\langle \varepsilon_i, \varepsilon_j \rangle = \langle \phi_i, \phi_j \rangle = 0; \qquad \langle \varepsilon_i, \phi_j \rangle = \delta_{ij}.$$

In constructing subdirect products in $A_1 \oplus A_2$, as in §3, where now $A_1 = H_1(\Sigma_g^+; \mathbb{Z}) \cong \mathbb{Z}^{2g}$, we take our "basepoint splitting" X of A_1 to be of the form $A_1 = M_X \oplus N_X$, where $\mathrm{Span}_{\mathbb{Z}}\{\varepsilon_1, \dots, \varepsilon_g\} \subset M_X$ and $N_X \subset \{\phi_1, \dots, \phi_g\}$. There is an infinite set of such splittings which we parametrise by suitable elements of the group $\mathrm{Sp}_{2g}(\mathbb{Z})$. With these modifications, we obtain the following analogue of Theorem 4.1.

Theorem 4.2. *Let Λ_1 be a surface group of genus $g \geq 2$, and let Λ_2 be a finitely generated group which maps epimorphically onto $\mathbb{Z}$; then there is a subset $\Theta \subset \mathrm{Sp}_{2g}(\mathbb{Z})$ which parametrises an infinite family $(\Gamma_\theta)_{\theta \in \Theta}$ of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$ with the property that $\Gamma_\theta \neq \Gamma_\sigma$ for $\theta \neq \sigma$.*

Theorem 4.3. *Let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, \langle\ ,\ \rangle)$ splits as an orthogonal direct sum*

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle),$$

where $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, and $\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$. Then there exists an infinite family $(\Gamma_\sigma)_{\sigma \in \Sigma}$ of mutually isomorphic finitely generated nonconjugate subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$.

Proof. $\mathrm{Aut}_{\mathbb{Z}}(L_i, \langle\ ,\ \rangle)$ is a finitely generated linear group, and so, by Selberg's Theorem, has a torsion free subgroup, Λ_i say, of finite index. Suppose that $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$; if $(L_1, \langle\ ,\ \rangle)$ is isotropic, then Λ_1 is free, whilst if $(L_1, \langle\ ,\ \rangle)$ is anisotropic, then Λ_1 is a surface group. Either way, if Λ_2 maps epimorphically onto $\mathbb{Z}$, we may apply the results of Theorems 4.1 and 4.2 to conclude that there is an infinite family, $(\Gamma_\delta)_{\theta \in \Theta}$, of mutually isomorphic finitely generated normal subdirect products of $\Lambda_1 \times \Lambda_2$. Moreover, since the family $(\Gamma_\theta)_{\theta \in \Theta}$ consists of *normal* subgroups of $\Lambda_1 \times \Lambda_2$, we see that no Γ_θ is conjugate in $\Lambda_1 \times \Lambda_2$ to any Γ_σ for $\theta \neq \sigma$.

Since $\Lambda_1 \times \Lambda_2$ has finite index in $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$, each Γ_θ is conjugate in $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ to at most finitely many Γ_σ. In particular, we may choose an infinite subfamily $(\Gamma_\sigma)_{\sigma \in \Sigma}$, so that no two distinct elements are conjugate in $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$. □

Although not conjugate in $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$, subgroups in the family $(\Gamma_\sigma)_{\sigma \in \Sigma}$ just constructed may become conjugate in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$. We show, however, that for each $\tau \in \Sigma$, the set $\{\sigma \in \Sigma : \Gamma_\sigma$ is conjugate to Γ_τ in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)\}$ is finite.

Theorem 4.4. *Let $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, \langle\ ,\ \rangle)$ splits as an orthogonal direct sum*

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle)$$

where $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, and $\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$. Then there exists an infinite family $(\Gamma_\omega)_{\omega \in \Omega}$ of mutually isomorphic finitely generated subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ such that Γ_ω is not conjugate, in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, to Γ_μ for $\omega \neq \mu$.

Proof. Let $\mathbb{G}$ (resp. $\mathbb{G}_i$) be the linear algebraic group whose group of k-rational points is $\mathrm{Aut}_k(L \otimes k, \langle\ ,\ \rangle)$ (resp. $\mathrm{Aut}_k(L_i \otimes k, \langle\ ,\ \rangle)$), and let $\mathbb{H} = \mathbb{G}_1 \times \mathbb{G}_2 \subset \mathbb{G}$. Let Γ_σ, Γ_τ be subgroups from the family constructed in Theorem 4.3, and suppose that for some $g \in \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, $g\Gamma_\sigma g^{-1} = \Gamma_\tau$. Since Γ_σ, Γ_τ are normal subdirect products of $\Lambda_1 \times \Lambda_2$, then by [3],

$$[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2] \subset \Gamma_\sigma \cap \Gamma_\tau .$$

Since $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, it follows that $\mathbb{G}_1$ is $\mathbb{Q}$-simple, and $\mathbb{G}_{1,\mathbb{R}}$ is noncompact. The condition that $\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$ implies that $(L_2, \langle\ ,\ \rangle)$ is indefinite, and that $\mathrm{rk}_{\mathbb{Z}}(L_2) \geq 3$. If $\mathrm{rk}_{\mathbb{Z}}(L_2) \neq 4$ then $\mathbb{G}_2$ is $\mathbb{Q}$-simple, and $\mathbb{G}_{2,\mathbb{R}}$ is noncompact. If $\mathrm{rk}_{\mathbb{Z}}(L_2) = 4$ then either $\mathbb{G}_2$ is $\mathbb{Q}$-simple, and $\mathbb{G}_{2,\mathbb{R}}$ is noncompact, or $\mathbb{G}_2$ is a product $\mathbb{L}_1 \times \mathbb{L}_2$ where $\mathbb{L}_1$ and $\mathbb{L}_2$ are both $\mathbb{Q}$-simple, and $\mathbb{L}_{1,\mathbb{R}}$, $\mathbb{L}_{2,\mathbb{R}}$ are both noncompact. Either way, if $\mathbb{L}$ is a $\mathbb{Q}$-simple factor of $\mathbb{G}_1 \times \mathbb{G}_2$, then $\mathbb{L}_{\mathbb{R}}$ is noncompact; applying (1.2) we conclude that $\overline{[\Lambda_i, \Lambda_i]} = \mathbb{G}_i$. Thus $\overline{[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2]} = \mathbb{H}$. It now follows that $g \in N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$.

Denote the index of $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ in $N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ by α. For each $\tau \in \Sigma$, the set $C_\tau = \{\sigma \in \Sigma : \Gamma_\sigma$ is conjugate to Γ_τ in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)\}$ has cardinality bounded by α. By (1.5), α is finite, so that each C_τ is finite. Let Ω be a subset of Σ obtained by choosing exactly one element from each C_τ; then Ω is infinite, and the family $(\Gamma_\omega)_{\omega \in \Omega}$ consists of isomorphic finitely generated subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, and has the desired property that Γ_ω is not conjugate, in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, to Γ_μ for $\omega \neq \mu$. □

Analogously, we show

Theorem 4.5. *Let* $\langle\ ,\ \rangle : L \times L \to \mathbb{Z}$ *be a nondegenerate symmetric bilinear form on a finitely generated free abelian group* L, *such that* $(L, \langle\ ,\ \rangle)$ *splits as an orthogonal direct sum*

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle) \perp (L_3, \langle\ ,\ \rangle)$$

where $(L_1, \langle\ ,\ \rangle)$ *has signature* $(2, 1)$, $\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ *has a subgroup of finite index which maps epimorphically onto* $\mathbb{Z}$, *and where* $(L_3, \langle\ ,\ \rangle)$ *is indefinite with* $\mathrm{rk}_{\mathbb{Z}}(L_3) \geq 3$. *Then there exists an infinite family* $(\Delta_\omega)_{\omega \in \Omega}$ *of mutually isomorphic finitely generated nonconjugate subgroups of* $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$.

Proof. Let $(\Gamma_\sigma)_{\sigma \in \Sigma}$ be the family of mutually isomorphic finitely generated nonconjugate subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ constructed in Theorem 4.3, and for each $\sigma \in \Sigma$, put

$$\Delta_\sigma = \Gamma_\sigma \times \mathrm{Aut}_{\mathbb{Z}}(L_3, \langle\ ,\ \rangle) \subset \mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_3, \langle\ ,\ \rangle).$$

Let $\mathbb{G}$ (resp. $\mathbb{G}_i$) be the linear algebraic group whose group of k-rational points is $\mathrm{Aut}_k(L \otimes k, \langle\ ,\ \rangle)$ (resp. $\mathrm{Aut}_k(L_i \otimes k, \langle\ ,\ \rangle)$), and let $\mathbb{H} = \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_3 \subset \mathbb{G}$. As in the proof of Theorem 4.4, we obtain $\overline{[\Lambda_1, \Lambda_1] \times [\Lambda_2, \Lambda_2]} = \mathbb{G}_1 \times \mathbb{G}_2$.

If $\mathrm{rk}_{\mathbb{Z}}(L_3) \neq 4$ then $\mathbb{G}_3$ is $\mathbb{Q}$-simple, and $\mathbb{G}_{3,\mathbb{R}}$ is noncompact. If $\mathrm{rk}_{\mathbb{Z}}(L_3) = 4$ then, since $(L_3, \langle\ ,\ \rangle)$ is indefinite, either the identity component of $\mathbb{G}_{3,\mathbb{R}}$ is isomorphic to $SO(3, 1)$ and $\mathbb{G}_3$ is $\mathbb{Q}$-simple, or $\mathbb{G}_{3,\mathbb{R}}$ is locally isomorphic to a product $SO(2, 1) \times SO(2, 1)$; either way, if $\mathbb{L}$ is a $\mathbb{Q}$-simple factor of $\mathbb{G}$, then $\mathbb{L}_{\mathbb{R}}$ is noncompact, so that we may apply Proposition 1.2 to conclude that $\overline{[\Lambda_3, \Lambda_3]} = \mathbb{G}_3$, and

$$\overline{[\Lambda_1, \Lambda_1]} \times \overline{[\Lambda_2, \Lambda_2]} \times \overline{[\Lambda_3, \Lambda_3]} = \mathbb{G}_1 \times \mathbb{G}_2 \times \mathbb{G}_3 .$$

As in the proof of Theorem 4.4, for each $\tau \in \Sigma$, the cardinality of the set

$$C_\tau = \{\sigma \in \Sigma \colon \Delta_\sigma \text{ is conjugate to } \Delta_\tau \text{ in } \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)\}$$

is bounded by the index, α, of $\mathrm{Aut}_{\mathbb{Z}}(L_1, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle) \times \mathrm{Aut}_{\mathbb{Z}}(L_3, \langle\ ,\ \rangle)$ in $N_{\mathbb{G}}(\mathbb{H}) \cap \mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$. By Proposition 1.5, α is finite, so that each C_τ is finite. Let Ω be a subset of Σ obtained by choosing exactly one element from each C_τ; then Ω is infinite, and the family $(\Delta_\omega)_{\omega \in \Omega}$ consists of isomorphic finitely generated subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, and has the desired property that Δ_ω is not conjugate, in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$, to Δ_μ for $\omega \neq \mu$. □

The referee points out that the condition "$\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ has a subgroup of finite index which maps epimorphically onto $\mathbb{Z}$", is precisely the same as requiring that $(L_2, \langle\ ,\ \rangle)$ have signature $(n, 1)$ for some $n \geq 2$. Indeed, if $\mathrm{Aut}_{\mathbb{Z}}(L_2, \langle\ ,\ \rangle)$ has a subgroup Γ of finite index which maps epimorphically onto $\mathbb{Z}$, then $H_1(\Gamma, \mathbb{Z})$ is infinite, so that, by Kazhdan's Theorem [4], $(L_2, \langle\ ,\ \rangle)$ has signature $(n, 1)$ for some $n \geq 2$. Conversely, Millson [6, §4] has shown that for any nondegenerate integral quadratic form $(L, \langle\ ,\ \rangle)$ of signature $(n, 1)$, with $n \geq 2$, there exists a subgroup Γ of finite index in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$ for which $H_1(\Gamma, \mathbb{Z})$ is infinite; in particular, Γ maps epimorphically onto $\mathbb{Z}$. Combining this observation with Theorems 4.4 and 4.5, we obtain

Corollary 4.6. *Let $\langle\ ,\ \rangle\colon L \times L \to \mathbb{Z}$ be a nondegenerate symmetric bilinear form on a finitely generated free abelian group L, such that $(L, \langle\ ,\ \rangle)$ splits as an orthogonal direct sum*

$$(L, \langle\ ,\ \rangle) \cong (L_1, \langle\ ,\ \rangle) \perp (L_2, \langle\ ,\ \rangle) \perp (L_3, \langle\ ,\ \rangle)$$

where $(L_1, \langle\ ,\ \rangle)$ has signature $(2, 1)$, $(L_2, \langle\ ,\ \rangle)$ has signature $(n, 1)$ for some $n \geq 2$, and either $L_3 = 0$ or $(L_3, \langle\ ,\ \rangle)$ is indefinite with $\mathrm{rk}_{\mathbb{Z}}(L_3) \geq 3$. Then there exists an infinite family $(\Delta_\omega)_{\omega \in \Omega}$ of mutually isomorphic finitely generated nonconjugate subgroups of $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$.

Our concern in this paper has been with conjugacy of subgroups within the discrete group $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$. From a different viewpoint, our results can be seen as a failure of the rigidity property for subgroups of infinite covolume within the corresponding Lie group $\mathrm{Aut}_{\mathbb{R}}(L \otimes_{\mathbb{Z}} \mathbb{R}, \langle\ ,\ \rangle \otimes 1)$; recall that the groups Γ_σ we construct all have infinite index in $\mathrm{Aut}_{\mathbb{Z}}(L, \langle\ ,\ \rangle)$. If G is a noncompact linear almost simple Lie group with $\mathrm{rank}_{\mathbb{R}} \geq 2$, then in consequence of the super-rigidity theorem of Margulis, when Δ is a discrete subgroup of finite covolume in G there are only finitely many G-conjugacy classes of discrete finitely generated subgroups isomorphic to Δ. The arguments of the present paper can be extended to show that under the hypothesis "Δ is finitely generated, discrete, of infinite covolume in G", the number of G-conjugacy classes of discrete finitely generated subgroups isomorphic to Δ becomes infinite in general. We will pursue this idea more fully elsewhere.

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